

Boundedness of moduli of Varieties of general type

- Preliminaries
 - Volume
 - Deformation invariance
 - DCC sets
 - Semi log canonical varieties.

Numerical dimension. $D_{\text{ps-FF}}$. $\dim X = n$.

$$K_6(x, D) := \max_{H \in \text{Pic}(X)} \left\{ k \in \mathbb{N} \mid \limsup_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X(mD \cap H))}{m^k} > 0 \right\}$$

D_{neF} .

$$K_6(x, D) = v(x, D) = \max \left\{ k \in \mathbb{N} \mid H^{n-k} \cdot D^k > 0 \right\}$$

If For example

$$K_6(x, D) = k(x, D), D : \underline{\text{sabundant}}$$

Volume $\dim X = n$,

$$\text{Vol}(X, D) := \limsup_m \frac{h^0(X, \mathcal{O}_X(mD))}{m^n}$$

restricted volume, $V \subset X$ $\dim V = d$

$\text{Vol}(X|V, D)$

$$:= \limsup_m \frac{d! \dim(\text{Im}(H^0(X, \mathcal{O}_X(mD)) \rightarrow H^0(\mathcal{I}_{(nD)_V}))}{m^d}$$

Lemma 2.2.2) $f: X \rightarrow Z$. b.r.

$(X, \Delta) (Z, B)$ l.c.

• $K_X + \Delta$ is b.g.

• (X, Δ) l.c. model $g: X \dashrightarrow Y$.

IF $F_* \Delta \leq B$ & $\text{vol}(X, K_X + \Delta) = \text{vol}(K_Z + B)$.

Then:

$Z \dashrightarrow X \dashrightarrow Y$
l.c model of (Z, B) .

Proof B_X strict trans of B on X .

F the F -exceptional divisors.

$\pi: W \rightarrow X$ log res of $(X, B_X + F)$
and of g .

$$K_W + \Theta = \pi^*(K_X + \Delta) + \underline{\underline{E}}$$

(W, Θ) . same l.c model as (X, Δ)
and same Volume.

replace (X, Δ) , by log smooth pair
 $g: X \rightarrow Y'$.

$$K_X + \underbrace{B_X + F}_D, \quad K_X + D = f^*(K_Z + B).$$

replace (Z, B) by (X, D) .

$$\text{vol}(K_X + \Delta) = \text{vol}(K_X + D).$$

$$D - \Delta \geq 0$$

!!

$$A := g_*(K_X + \Delta) \quad \underline{\text{ample.}}$$

$$H := g^*(A), \quad K_X + \Delta - H \geq 0.$$

Claim
 $D - \Delta$ is g-exceptional.

S a component in L w/ $\text{coeff } a > 0$

$$V(t) = \text{Vol}(X, H + tS).$$

$$\begin{aligned}
 \underline{\underline{V(O)}} &= \underline{\underline{Vol(X, H)}} \\
 &= \underline{\underline{Vol(X, K_x + D)}} \\
 &= \underline{\underline{Vol(K_x + D)}} \\
 &\geq \underline{\underline{Vol(H + L)}}. \\
 &\geq \underline{\underline{Vol(H + \alpha S)}} \\
 &= \underline{\underline{V(a)}}.
 \end{aligned}$$

$$K_x + D \geq H + L$$

↔

V is constant near O .

$$\begin{aligned}
 0: \frac{1}{n} \frac{dV}{dt} \Big|_{t=0} &= \underline{\underline{Vol_{X \subseteq}(H)}} \geq \underline{\underline{S \cdot H}}^{n-1} \\
 &= g \cdot S \cdot \underline{\underline{A}}^{n-1}
 \end{aligned}$$

$$g \cdot S = 0 \Rightarrow L \text{ is } \underline{\underline{g\text{-exceptional}}}.$$

Deformation ; h variance.

Lemma. 2.3.1] $\gamma: X \rightarrow U$.

(X, Δ) log smooth over U .

A rel. ample. $(X, \Delta + A')$ is log sm. over U .

$$A' \sim \lfloor \Delta \rfloor + A_\circ$$

Then:

$$f_* (\mathcal{O}_X(m(K_X + \Delta) + A))$$

$$\rightarrow H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u) + A_u))$$

$\forall u \in U$.

Proof.

$$m(K_X + \Delta) + A \sim m(K_X + \Delta - \frac{1}{m} \lfloor \Delta \rfloor + \frac{1}{m} A')$$

$$\begin{matrix} !! \\ \Delta \\ = \end{matrix}$$

$$\Delta \leq \Delta' \leq \Delta + A'. \quad (X, \Delta') \text{ is log smooth}$$

with $\Delta' < 1$. Δ' is big over.

$$D' \sim D + \frac{1}{m} \underbrace{A}_{\text{ample.}}$$

HMX. (1.8.1).

$$h^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + D_u'))$$

are independent of u .

so the rest. \leq scerj. for all u_0

Def. $\tau: X \rightarrow U$ D div. C prime divisor

For D big.

$$\begin{aligned} \sigma_C(X/U, D) &:= \\ &\inf \left\{ \text{mult}_C(D') \mid D' \sim_{R_U} D, D' \geq 0 \right\} \end{aligned}$$

D is pseudo-effective.

$$\sigma_C(X/U, D) := \lim_{\epsilon \rightarrow 0} \sigma_C(X/U, D + \underline{\epsilon} A)$$

$$N_0(X/U, D) = \sum_C \delta_C(X/U, D) C$$

Lemma 2.3.2]

$\pi: X \rightarrow U$. (X, Δ) log smooth.

$K_X + \Delta$ ps-eff.

$$N_0(X/U, K_X + \Delta) \Big|_{X_u} = N_0(X_u, K_{X_u} + \Delta_u)$$

$u \in U$.

$$\text{Proof } (N_0) \Big|_{X_u} \geq N_0(X_u, K_{X_u} + \Delta_u)$$

By definition..

Pick $A, [\Delta] + A \sim A'$

with $(X, \Delta + A')$ log smooth.

$$f_* \left(\mathcal{O}_X(m(K_X + \Delta) + A) \right) \rightarrow H^0(X_u, \dots)$$

$$\Rightarrow (N_6) \Big|_{X_u} \leq (N_6(X_u))$$



[Lemma 2.3.3] $\pi: X \rightarrow U$

(X, Δ) log smooth over U_0

strata of Δ have irreducible fibers.

$K_X + \Delta$ ps-efF. $0 \in U$ closed.

$$\Theta_0 := \underline{\Delta_0 - \Delta_0 \cap N_6(K_0, K_{x_0} + \Delta_0)}$$

$$0 \leq \Theta \leq \Delta \text{ s.t. } \Theta_0 = \Theta|_{X_0}$$

Then:

$$\underline{\Theta} = \left(\Delta - \Delta \cap N_6(X/U, K_X + \Delta) \right)_0$$

Proof:; Pick ample H .
 $0 \leq S \leq \lfloor \Delta \rfloor$. exists

$$H' \sim H + S.$$

Set $(X, \Delta + H')$ is log smooth

Fix $m \in \mathbb{N}$.

$$\phi_0 := \Delta_0 - \Delta_0 \text{NN}(x_0, K_{x_0} + \Delta_0 + \frac{1}{m} H_0)$$

$$\boxed{\phi_0 = \phi|_{x_0}} \quad \boxed{0 \leq \phi \leq \Delta}$$

$$\pi_* \mathcal{O}_X(m(K_X + \phi) + H) \xrightarrow{\text{inj.}} \pi_*(m(K_X + \Delta) + H)$$

Surj.

Surj.

$$H^0(C, (m(K_{x_0} + \phi_0) + H_0)) \xrightarrow[\text{bij.}]{} H^0(m(K_{x_0} + \Delta_0) + H_0)$$

$$\phi \leq \Delta$$

We can locally lift the surjection.

$$\pi_* (\mathcal{O}_X(m(K_X + \phi) + H)) \rightarrow$$

$$\pi_* (\mathcal{O}_X(m(K_X + \Delta) + H))$$

$$m(K_X + \phi) + H \geq m(K_X + \Delta) + H$$

$$-N_6(m(K_X + \Delta)) + H - \frac{H}{m}$$

$$\boxed{\phi \geq \Delta - N_6(K_X + \Delta + \frac{1}{m}H)}$$

$$\phi \geq \Delta - \Delta \wedge$$

$$\theta \geq \Delta - \Delta \wedge N_6(K_X + \Delta)$$

$$(\underline{A \wedge B})|_{x_0} \leq \underline{A}|_{x_0} \wedge \underline{B}|_{x_0}$$

$$(\Delta - \Delta \wedge N_6)|_{x_0} = \Delta_0 - (\Delta \wedge N_6)|_{x_0}$$

$$\geq \Delta_0 - \Delta_0 \wedge N_6 (k_{x_0} + \Delta_0)$$

$$\Rightarrow \theta_0 \leq (\Delta - \Delta \wedge N_6)|_{x_0}$$

by uniqueness

$$\boxed{\Rightarrow} \quad \theta \leq \Delta - \Delta \wedge N_6,$$

□

Lemma 2.3.4] $\pi: X \rightarrow U$.

(X, D) log smooth

$\text{coeff}(D) = 1$. $0 \in U$ closed point.

$$\pi_* \mathcal{O}_X(K_X + D) \rightarrow H^0(X_0, \mathcal{O}_X(K_{X_0} + D_0))$$

Proof: We can cut by hyperplanes

assume U is a curve:

We want:

$$H^0(X, \mathcal{O}_X(K_X + X_0 + D) \rightarrow H^0(X_0, \mathcal{O}_X(K_{X_0} + D_0))$$

is Surj.

$$0 \rightarrow \mathcal{O}_X(-X_0) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{X_0} \rightarrow 0.$$

$$\otimes (K_X + K_0 + D)$$

$$\rightarrow H^0(\mathcal{O}_X(K_X + \underline{x_0} + D) \xrightarrow{\text{surj.}} H^0(\mathcal{O}_{x_0}(k_{x_0} + D))$$

$\hookrightarrow H^1(\mathcal{O}_X(K_X + D)) \xrightarrow{\text{inj.}} H^1(\mathcal{O}_X(K_X + \underline{x_0} + D))$

We want injectivity.

Kollar's injectivity Lemma

$$H^1(X, \mathcal{O}_X(K_X + \underbrace{B + \sqrt{m}H}_{m>0})) \rightarrow H^1(\mathcal{O}_X(K_X + B + (\sqrt{m} + m)H))$$

DCC sets

Lemma 2.4.1 $\underline{\mathbb{I}} \subseteq \mathbb{R}$ DCC.

Fix d .

$$T_d := \left\{ \left\{ d_1, \dots, d_k \right\} \mid k \in \mathbb{N}, d_i \in \mathbb{I} \right. \\ \left. \sum d_i = d \right\}$$

T_d is finite.

Proof $k < N$, constant number

of elements., we take non-dec.
 Sequences of k -tuples.
 subsequence has to be constant.

$\rightarrow \underline{\text{Finite}}$

Lemma 2.4.2]

$J \subseteq \mathbb{R}_{\leq 1}$ finite.

$$J^0 = \left\{ \alpha \in [0, 1] \mid \alpha = 1 + \sum_{i=1}^k \alpha_i \cdot 10^{-i} \right\}$$

$\alpha_i \in J$.

is Finite.

Proof $\alpha_i \neq 1$, $\alpha_i \leq 1 - \varepsilon$.

$$\Rightarrow \alpha = 1 + \sum \alpha_i \cdot 10^{-i} \geq 1 + \sum (\alpha_i - 1) \geq 0.$$

$$K \leq \frac{1}{\epsilon}.$$

→ Finite possibilities



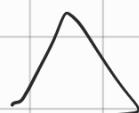
Semi log canonical varieties.

X demi-normal.

• S_2 .

• in codimension 1, singularities
are double normal crossings.

X dem-normal



$$K_X + \underline{\Delta}$$

$n: Y \rightarrow X$ normalization.

$$K_Y + \underline{\Gamma} = n^*(K_X + \underline{\Delta})$$

(X, Δ) is slc. if (Y, Γ)
is lc.

Theorem 2.5.1]

(X, Δ) sl.c. $n: Y \rightarrow X$,

$$K_Y + \Gamma = n^*(K_X + \Delta).$$

(Y, Γ) l.c.

$K_X + \Delta$ is semi-ample iff

$K_Y + \Gamma$ is semi-ample.

Proof] $K_X + \Delta$ is semi-ample,

then so is $K_Y + \Gamma$.

Sketch: $K_Y + \Gamma$ semi-ample.

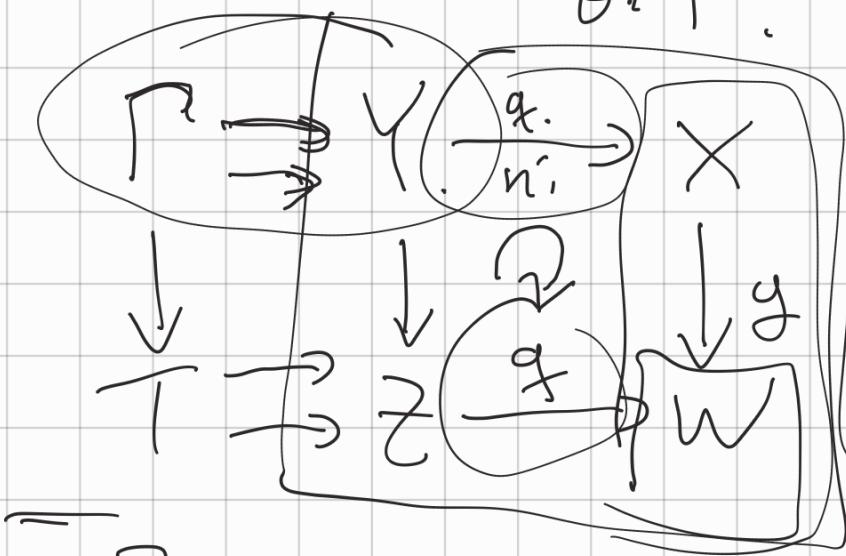
$$m(K_Y + \Gamma) = : M : \text{ sb. p. F.}$$

$$g: Y \rightarrow Z$$

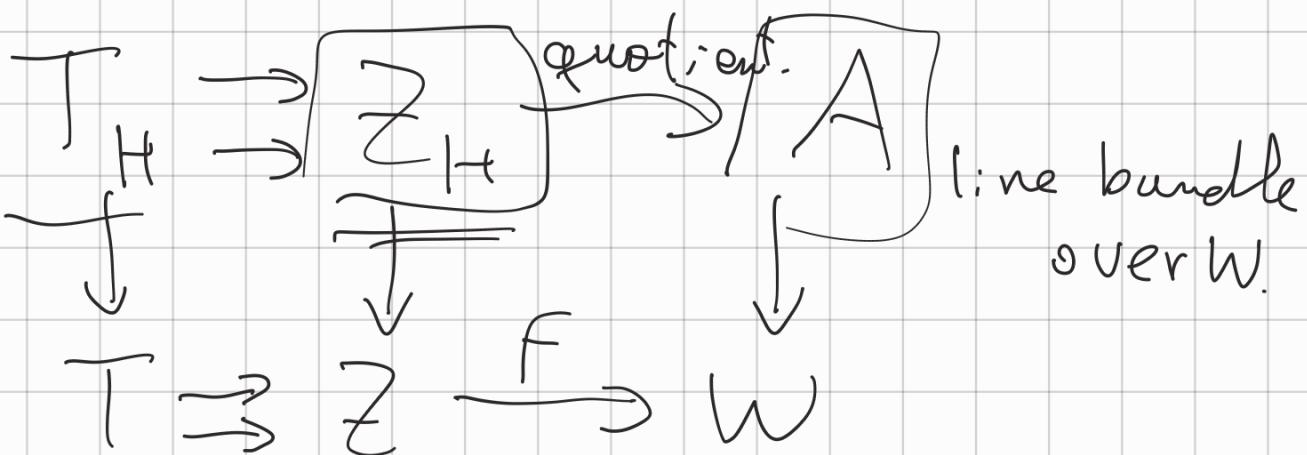
$$g^*(H) = M \quad H \text{ very ample on } Z.$$

double locus on Y has an involution

on Γ .



Z_H, T_H to be total spaces
of H and $H|_{T_H}$.



$$f^*(A) = \mathcal{H}$$

$$g^*(A) = \mathcal{O}_x(m(K_x + \Delta))$$

$K_x + \Delta$ is semi-ample.

□

Fim.